University of North Georgia
Department of Mathematics

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Course: College Algebra Math 1111

Text Book: For this course we use the free e – book by Stitz and Zeager with link:
http://www.stitz-zeager.com/szca07042013.pdf

Tutorials and Practice Exercises

- http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/index.htm
- http://www.mathwarehouse.com/algebra/
- http://www.ixl.com/math/algebra-2
- http://www.ixl.com/math/precalculus
- http://www.ltcconline.net/greenl/java/index.html

For more free supportive educational resources consult the syllabus

Chapter 3

Polynomial Functions (page 235)

3.1 Polynomial Functions and Their Graphs

Objectives: By the end of this section students should be able to:

- **Identify polynomial Functions**
- **Graph basic polynomial functions**
- **Identify end behavior and leading term**
- **Use zeros in graphing polynomials**
- **Find local maximum and local minimum**

Definition of a Polynomial Function

A polynomial function of degree n is a function of the form

$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is a **non-negative integer** and $a_n \neq 0$

- The numbers $a_n,\ a_{n-1},\ ...\ a_1,\ a_0$ are called the **coefficients** of the polynomial
- The number a_0 is called the **constant coefficient** or **constant term**
- The number a_n, the coefficient of the highest power, is the **leading coefficient**
- The term a_nx^n is the **leading term**
- $p(x) = a_0$ is a polynomial of **degree 0**
- If $p(x) = 0$, we say **P** has **no degree**

Example 3.1.1 Page 235 reading: Which of the following functions are polynomials/

Example 1: Which of the following is a polynomial? If an expression is a polynomial, **name its degree**, and tell the variable that the polynomial is in.

a) $f(x) = x^3 - 2x^2 - 3x - 4$

b) $p(y) = 3y^2 + 2y + 1$

c) $h(x) = x3 + 2\sqrt{x} + 1$

d) $f(x) = \sqrt[3]{x^2}$

e) $f(z) = z^{-1} + 2$

Example 2: $p(x) = -2x^5 + x^4 + 3x^3 - 4x^2 - 12x + 6$

a) Degree
b) Leading term
c) Leading coefficient
d) Constant term

Example 3.1.2 Page 237 Reading

Example 3: Give examples of polynomial function of

a) degree 0
b) degree 1
c) degree 2
d) degree 33

Example 3.1.3 Page 338: An open box will be made by cutting out congruent squares from each corner of a 3ft by 8ft piece of cardboard and then folding up the sides. Let x denote the length of the side of the square which is removed from each corner.

a) Find the volume V of the box as a function of x. Include appropriate applied domain

b) Using graphing calculator graph $y = \mathrm{V}(x)$ on the domain from 1) and approximate the maximum volume to two decimal places. What is the maximum volume?

Graphs of Basic Polynomials

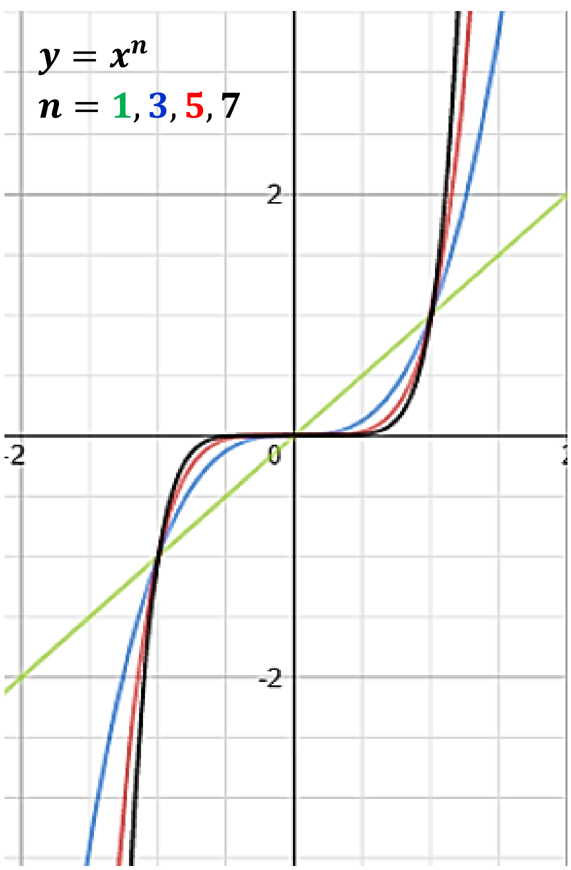
The simplest polynomial functions are the monomials $\boldsymbol{p(x) = ax^n, a \neq 0}$

Example 4: Using graphing utilities sketch the graph of $\boldsymbol{y = ax^n}$ **,** for $\boldsymbol{n = 1, 2, 3, 4, 5}$ **and so on** and deduce some general properties of the graphs of $\boldsymbol{y = ax^n}$

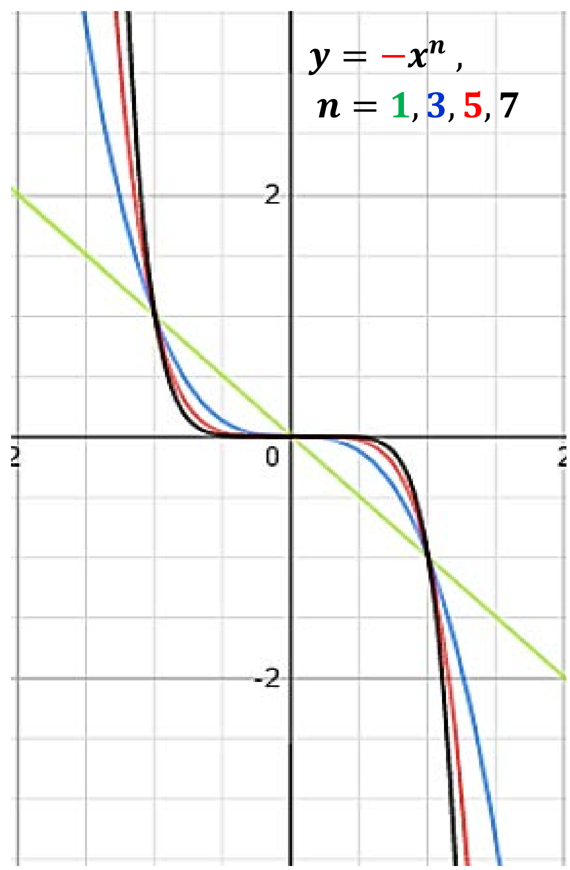
Recall: The coefficient $\boldsymbol{a}$ in $\boldsymbol{y = ax^n}$, for $\boldsymbol{a > 0}$ either **stretches** or **shrinks** the graph vertically. If $\boldsymbol{a < 0}$ the graph of $\boldsymbol{y = ax^n}$ is a **reflection** of $\boldsymbol{y = |a|x^n}$ across the x-axis. Therefore, we consider graphs of $\boldsymbol{y = ax^n}$ for the cases where $\boldsymbol{a = \pm 1}$. **Two cases** $\boldsymbol{i)\ n - odd\ and\ ii)\ n - even}$

i) $\boldsymbol{n - odd;\ n = 1, 3, 5, 7}$ ...

a) $\boldsymbol{a > 0}$ End behaviour**: DOWN, UP**



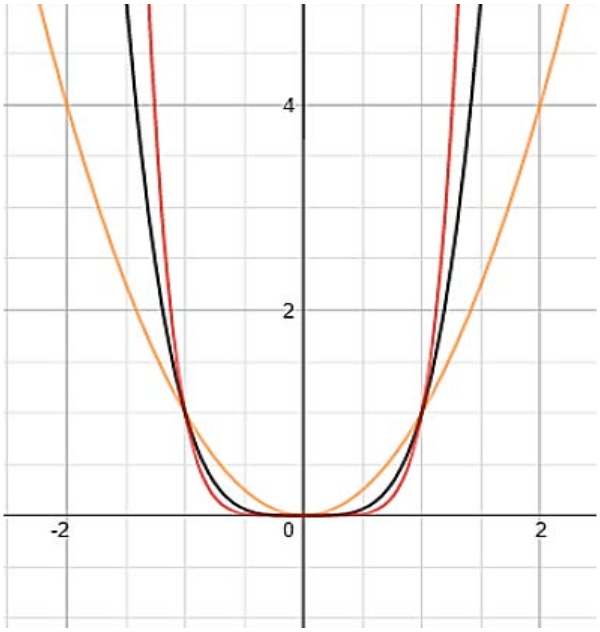
b) $\boldsymbol{a < 0}$ End behaviour**: UP, DOWN**



ii) $n - even;\ n = 2, 4, 6,\ 8 \ldots$

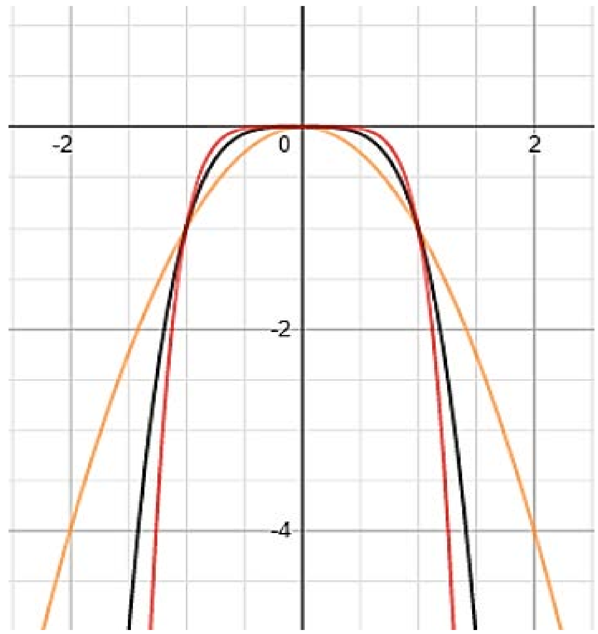
a) $a > 0$ End behaviour: **UP, UP**

$y = x^n, n = 2, 4, 6$



b) $a < 0$ End behaviour: **DOWN, DOWN**

$y = -x^n, n = 2, 4, 6$



Let **P** be a polynomial function of **degree n.**

1. The domain of **P** (any polynomial function) is the **set of all real numbers**
2. **P** is continuous for all real numbers, so there are no **breaks**, **holes**, or **jumps** in the **graph**.
3. The graph of **P** is a **smooth curve** with rounded corners and no sharp corners or cusps.
4. The graph of **P** has **at most** ***n x-intercepts*** **or** ***n-zeroes***.
5. The graph of **P** has **at most n – 1 turning points**.
6. The graph of **P** has f**our** types of **end behaviour**s

Example 5: Sketch the graph of the following functions using graphing utilities and find their end behavior, number of zeros and turning points

a) $f(x) = x^3 - 2x^2 - 3x + 2$

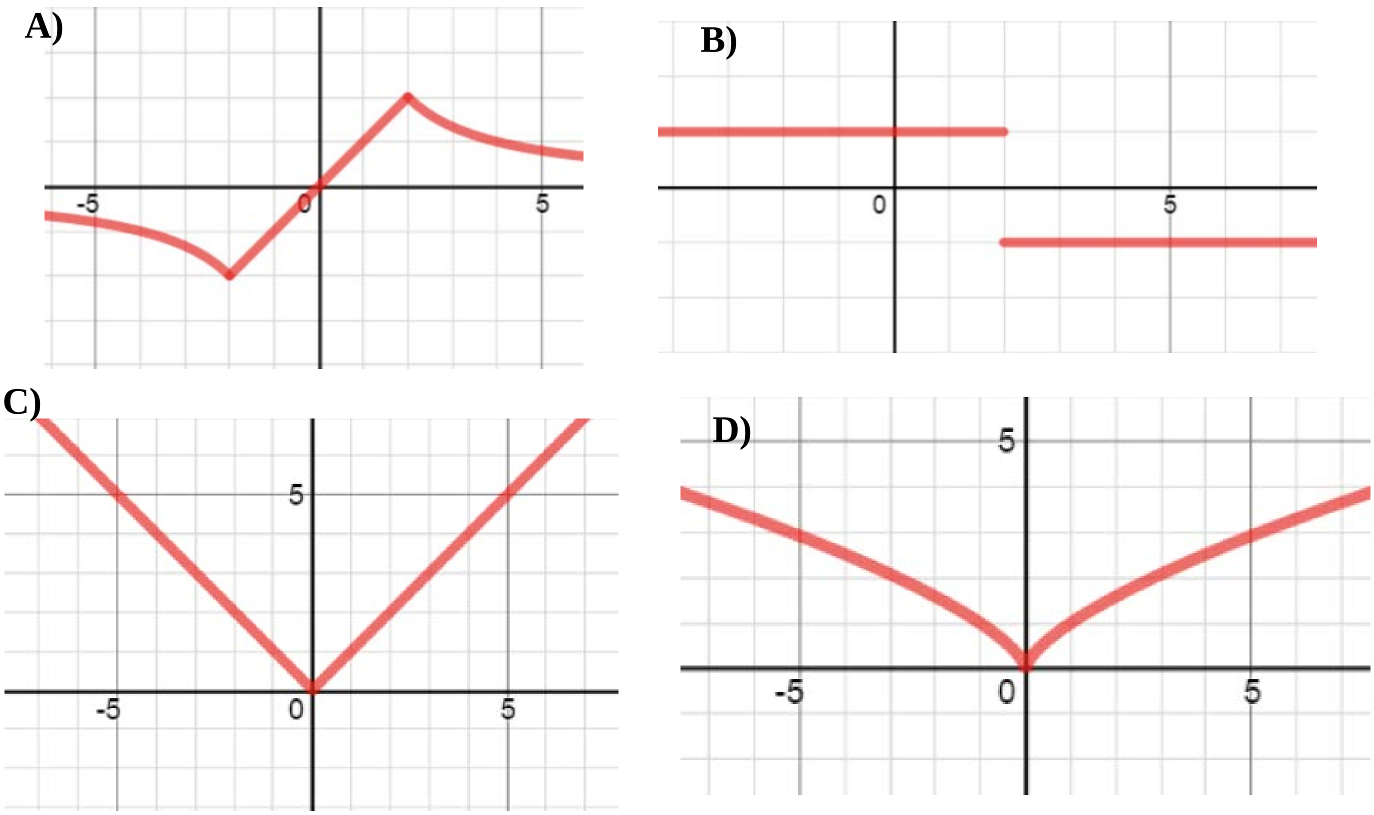
b) $f(x) = -x^3 - 2x^2 + 3x + 2$

c) $f(x) = -x^4 + 4x^2 - 1$

d) $f(x) = x^4 - 4x^2 + 1$

e) $P(x) = x^2(x+2)(x+1)(x-1)(x-2)$

Example 6: Each of the following graphs **cannot** be a graph of a polynomial function; Why?.



Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a **polynomial function of degree n**

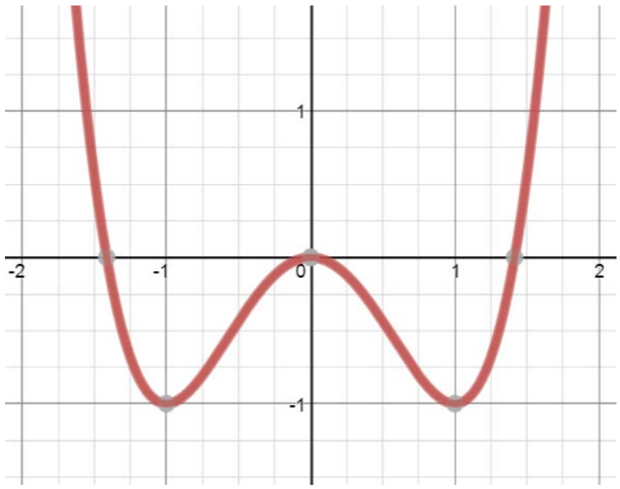
Table 1 shows **End Behaviors** for the graphs of **polynomial functions** of **Degree** n and **Leading Coefficient** a_n

Leading Coefficient / Degree	**End Behavior**
$a_n > 0$ n **even**	**both ends up** **(up, up)**
$a_n < 0$ n **even**	**both ends down** **(down, down)**
$a_n > 0$ n **odd**	**left down, right up** **(down, up)**
$a_n < 0$ n **odd**	**left up, right down** **(up, down)**

Table 1

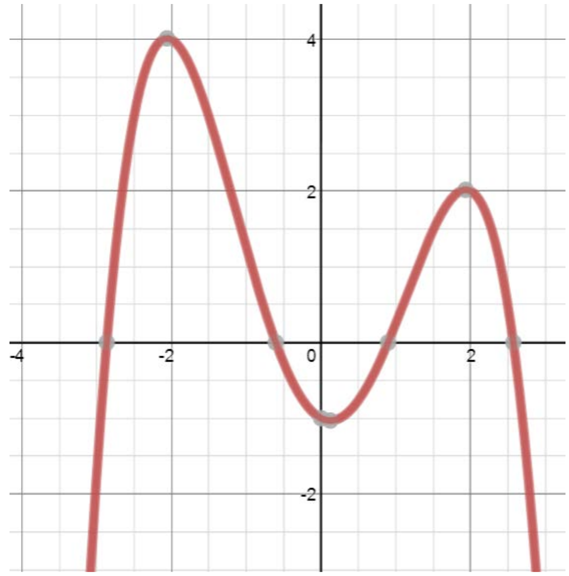
Example 7: Classifying Polynomials by Their Graphs

i. Even Degree



$y = x^4 - 2x^2$

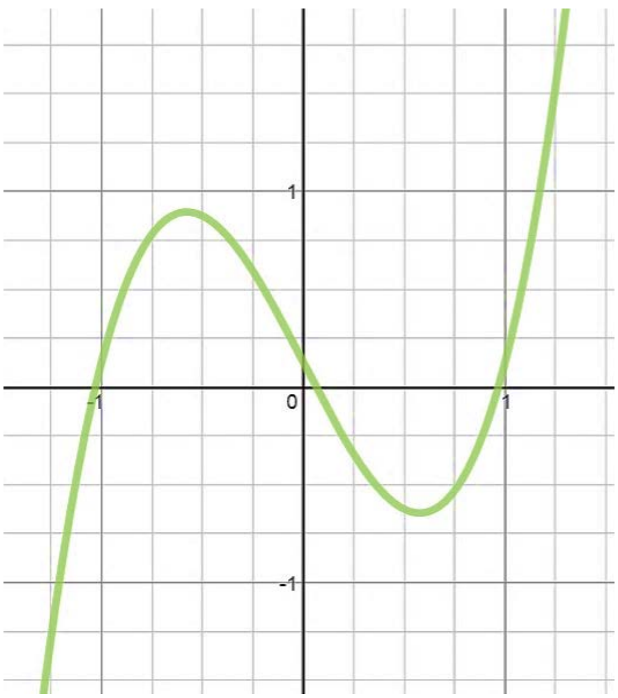
$a_n > 0$, **opens up or up, up**



$y = -\frac{1}{4}x^4 + 2x^2 - \frac{1}{2}x - 1$

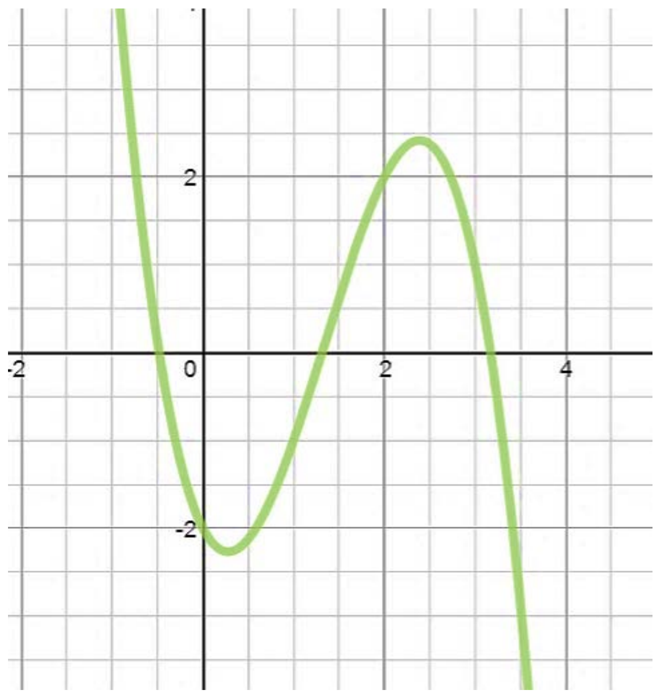
$a_n < 0$, **opens down or down, down**

ii. Odd Degree



$y = 2x^3 - 2x + \frac{1}{8}$

$a_n > 0$; **Left Down, Right UP**



$y = -x^3 + 4x^2 - 2x - 2$

$a_n < 0$; **Left UP, Right Down**

Example 8: Determine the **leading term**, the **leading coefficient**, the **degree** of the **polynomial**, and the **end behavior** of the graph.

a) $f(x) = 2x^3 + 3x^2 - 5x + 4$

Ans. 1) leading term = $\mathbf{2x^3}$

2) Leading coeff. = **2**

3) Degree = **3**

4) End Behavior **down, up**

b) $f(x) = -x^4 + 2x^3 + 3$

Ans. 1) Leading term = $\mathbf{-x^4}$

2) Leading coeff. = **–1**

3) degree = **4**

4) End Behavior: **down, down**

c) $f(x) = -x^5 + 3x^3 + 7$

d) $f(x) = 2x^2 + 3x + 2$

Example: Name the **degree**, the **leading coefficient**, and the **constant term** of

$$\boldsymbol{h(x) = (5x + 1)(3x - 1)(2x + 5)^3}$$

Solution:

If we were to multiply out, then the **degree** of the **product** would be the **sum** of the **degrees** of **each factor**, thus the degree of $\boldsymbol{h(x)} = \mathbf{1 + 1 + 3 = 5}$.

Note: $h(x) = (5x + 1)(3x - 1)(2x + 5)^3 = (5x + 1)(3x - 1)(2x + 5)(2x + 5)(2x + 5)$.

The **leading coefficient** would be the **product** of all the **leading coefficients**: $5 \cdot 3 \cdot 2^3 = 15 \cdot 8 = 120$.

And the **constant term** would be the **product** of all the **constant terms**: $1 \cdot (-1) \cdot 5^3 = -1 \cdot 125 = -125$.

Example 9: Find the degree, the leading coefficient, and the constant term.

a) $\boldsymbol{f(x) = 6x^3 + 7x^2 - 3x + 1}$

b) $\boldsymbol{f(x) = (x - 1)(x^2 + x - 6)}$

b) $\boldsymbol{g(x) = (x + 2)^2(x - 3)^3(2x + 1)^4}$

c) $\boldsymbol{h(x) = x(x - 2)^5(x + 3)^2}$

d) $\boldsymbol{f(x) = 5x^3 - 4x^2 + 7x - 8}$

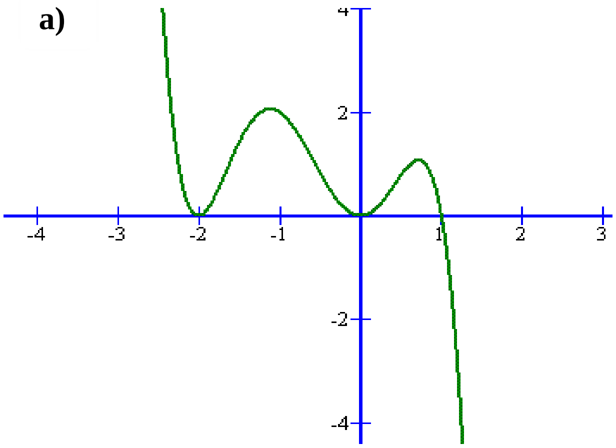
Example 10: Referring to the graphs below:

i. Identify as **even** or **odd** degree **polynomials**

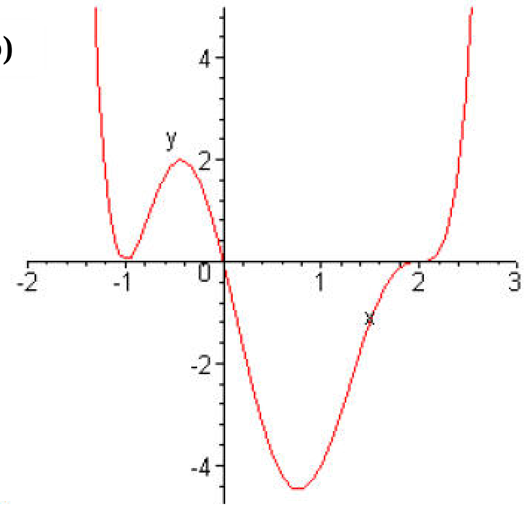
ii. Determine **possible degrees** and **signs of leading coefficients**

iii. Find possible **zeros** of the **polynomials.**

a)



b)



The zeroes of a polynomial $y = p(x)$ and Multiplicity

Zeroes of a polynomial

Recall: If $\boldsymbol{p(r) = 0}$ for a number $\boldsymbol{r}$, then $\boldsymbol{r}$ is called the **zero** of $\boldsymbol{p}$, to find the zeros of $\boldsymbol{p}$:

- Set $\boldsymbol{p(x) = 0}$ and solve for $\boldsymbol{x}$.
- Factor, if it is possible to factor, the polynomial $\boldsymbol{p}$

Example 11: Find the zeros of $\boldsymbol{f(x) = x^3 + 2x^2 - 5x - 6}$.

Factor $\boldsymbol{f(x)}$ and set it **equal to 0** and **solve** for **x.**

$\boldsymbol{x^3 + 2x^2 - 5x - 6} = (x + 3)(x + 1)(x - 2)$

$x = -3, -1,\ and\ 2$, are the zeros of the function.

Note that: $f(-3) = 0,\ f(-1) = 0, and\ f(2) = 0$

Multiplicity

Definition (Multiplicity)

The **multiplicity** of a zero is the number of times that zero occurs. For the polynomial function $\boldsymbol{f(x) = (x - c)^k}$, $\boldsymbol{c}$ is a **zero** of the function with **multiplicity** $\boldsymbol{k}$.

- If **k** is **odd**, then the graph **crosses** the x-axis at **(c, 0)**
- If **k** is **even**, then the graph **is tangent to** the x-axis at **(c, 0)** (touches the x-axis but does not cross it)

Theorem: Suppose **P** is a polynomial function and $x = c$ is a zero of multiplicity m. Then:

- If **m** is even, then the graph of **P is tangent to** the x-axis at **(c, 0)** (touches and re-bounce from the x-axis at **(c, 0)**)
- If m is **odd**, then the graph of **P crosses** the *x*-axis at **(c, 0)**

Example 14: For each of the following find the zeroes, state the multiplicity, and sketch the graph

a) $f(x) = 5x(x-2)^3(x+1)$

Solution: $x = 2$ is a zero with ***multiplicity 3***; *(graph crosses the at $x = 2$)*
$x = -1$ is a zero with **multiplicity *1;*** *(graph crosses the at $x = -1$)*
$x = 0$ is a zero with multiplicity 1; *(graph crosses the at $x = 0$)*

b) $f(x) = -x^2(x-1)^3(x+2)^4$.

Solution: $x = -2$ is a zero with ***multiplicity 4***; *(graph re-bounces at $x = -2$)*
$x = 1$ is a zero with **multiplicity *3;*** *(graph crosses the at $x = 1$)*
$x = 0$ is a zero with multiplicity 2; *(graph re-bounces at $x = 0$)*

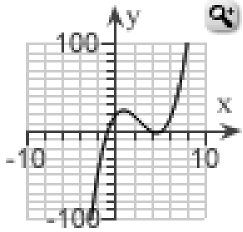
Example 3.1.6 page 245: Reading

Important ideas for sketching graphs of polynomials:

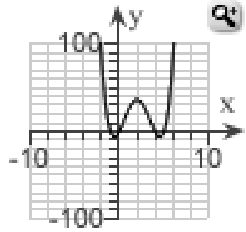
- **Zeros and their multiplicity**
- **Degree and leading coefficient**
- **End Behavior:** The **leading term** and the **degree** tells us about the **end behavior**
- **Intercepts: x and y - intercepts**
- **Symmetries:** If any
- **Test Points:** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept on the table.
- **Graph:** Plot the intercept and other points you found on the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

Example 12: Choose the correct graph of $\boldsymbol{h(x) = -x(x-4)(x+1)(x-5)}$

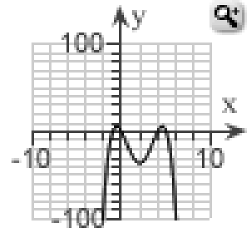
A.



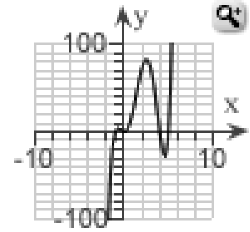
B.



C.

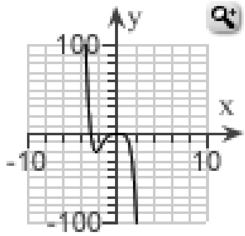


D.

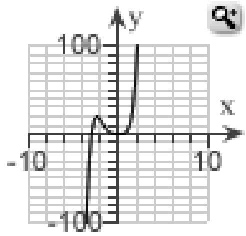


Example 13: Choose the correct graph of $\boldsymbol{f(x) = x^5 + 3x^2}$

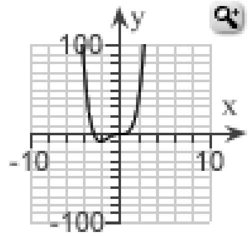
A.



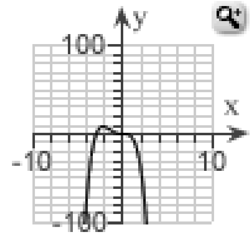
B.



C.



D.



Example 15: Sketch the graph of the following polynomials

a) $f(x) = 6x^3 + 7x^2 - 3x + 1$

b) $f(x) = (x - 1)(x^2 + x - 6)$

c) $g(x) = x(x + 2)(x - 3)^2(2x + 1)$

d) $h(x) = -(x-1)(x-3)(x+2)(x+1)$

OER 1: West Texas A&M University Tutorial 35: Graphs of Polynomial Functions

Practice Problems from the Text
Page 246, Exercises 3.1.1: #1 – 32 (odd numbers)

Polynomial Divisions (Page 257)

Long Division and Synthetic Division

Long Division

The Division Algorithm

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there are **unique polynomials** $Q(x)$ and $R(x)$, where $R(x)$ is **either 0** or of **degree less** than the **degree** of $D(x)$, such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

The polynomials $P(x)$ and $D(x)$ are called the **Dividend** and **divisor respectively**

$Q(x)$ **is** called the **quotient**

$R(x)$ is called the **remainder**

For Example: If we divide $6x^2 - 26x + 12$ *by* $x - 4$ we get

$$6x^2 - 26x + 12 = (x - 4)(6x - 2) + 4$$

In the **Division Algorithm** Format:

$P(x) = 6x^2 - 26x + 12$ is the **Dividend;**

$D(x) = x - 4$ is the **Divisor**;

$Q(x) = 6x - 2$ is the **Quotient** and

$R(x) = 4$ is the **Remainder**

Example 1: Divide $(x^4 - 2x^2 + x - 2) \div (x^2 + x - 4)$

Solution: By **Division Algorithm**:

$$x^4 - 2x^2 + x - 2 = (x^2 + x - 4) \cdot Q(x) + R(x)$$

Where $Q(x)$and $R(x)$ **are polynomials** to be determined using **Polynomial long Division**

In dividing polynomials using **Long Division**:
First we **must insert zero** placeholders for missing terms and rewrite the division as:

$$(x^4 + 0x^3 - 2x^2 + x - 2) \div (x^2 + x - 4)$$

Next, set up the polynomial division as a standard division problem and repeat the steps Divide, Multiply, Subtract, Carry Down over and over until the divisor no longer may be divided into the result at the bottom.

Step 1: We eliminate $\boldsymbol{x^4}$ from the **dividend**, to do so we need to multiply the **divisor** by x^2 and subtract the product from the dividend and bring down $\boldsymbol{x}$ to get a **new dividend,** $\boldsymbol{-x^3 + 2x^2 + x}$

Step 2: Next we eliminate $\boldsymbol{-x^3}$ from **the new dividend,** to do so multiply the **divisor** by $\boldsymbol{-x}$ and subtract the product from $\boldsymbol{-x^3 + 2x^2 + x}$ and bring down $\boldsymbol{-2}$, which gives a **second new dividend** $\boldsymbol{3x^2 - 3x - 2.}$ Repeat **this process** for the **new dividend,** until we get a **dividend** of **degree smaller** than the **divisor,** $\boldsymbol{x^2 + x - 4}$

$$
\begin{array}{r|l}
 & x^2 - x + 3 \\
\hline
x^2 + x - 4 & x^4 + 0x^3 - 2x^2 + x - 2 \\
 & \underline{x^4 + \;\; x^3 - 4x^2} \\
 & 0 - \;\; x^3 + 2x^2 + x \\
 & \quad \underline{- x^3 - \;\; x^2 + 4x} \\
 & \qquad 0 \; + 3x^2 - 3x - 2 \\
 & \qquad\quad \underline{3x^2 + 3x - 12} \\
 & \qquad\qquad - 6x + 10
\end{array}
$$

Since, **– 6x + 10** is of smaller degree than $\boldsymbol{x^2 + x - 4}$**,** we stop the process here.
The polynomial $\boldsymbol{x^2 - x + 3}$ is **Quotient,** and $\boldsymbol{-6x + 10}$ is **reminder; and so,**

$$x^4 - 2x^2 + x - 2 = (x^2 + x - 4) \cdot (x^2 - x + 3) + (-6x + 10)$$

Example 2: Using Long Division, find the quotient and the remainder of each of the following.

a) $f(x) = \frac{3x^3+2x-4}{x^2-4}$

b) $f(x) = \frac{5x^4+3x^2+2x-8}{2x^2+2x-8}$

c) $f(x) = \frac{x^4 + 2x^3 - 6x^2 - 2}{x - 1}$

OER: West Texas A&M University Tutorial 36: Long Division

Homework Practice Problems from the Text
Exercise 3.2.1 Page 265 #1 – 6

Synthetic Division - The Shortcut for Dividing by $(x - c)$

When dividing a polynomial $f(x)$ by a linear factor $(x - c)$, we can use a shorthand notation saving steps and space.

Procedure for Synthetic Division; we proceed with example

Example 3: Divided $f(x) = 3x^3 + 2x - 1$ by $(x - 4)$.

1. Insert **zero** place holder for the missing term: $f(x) = 3x^3 + 0x^2 + 2x - 1$
2. Write the value of "**c**" and the coefficients of $f(x)$ in a row. In **Example 3** **c = 4**, and the coefficients are **3, 0, 2,** and **-1.**

$$\begin{array}{r|rrrr} 4 & 3 & 0 & 2 & -1 \\ & & & & \\ \hline \end{array}$$

3. Carry down the first coefficient. In this case carry down the **3**.

$$\begin{array}{r|rrrr} 4 & 3 & 0 & 2 & -1 \\ & & & & \\ \hline & 3 & & & \end{array}$$

4. Multiply this carried down coefficient by the value of c.
   In this case, multiply **3 • 4 = 12**. Place this result in the next column.

$$\begin{array}{r|rrrr} 4 & 3 & 0 & 2 & -1 \\ & & 12 & & \\ \hline & 3 & & & \end{array}$$

5. Add the column entries and place result at bottom. In this case you add **0 + 12** to get **12.** Multiply this addition result by "c" and place in next column. In this case you multiply **12 • 4 = 48**

$$\begin{array}{r|rrrr} 4 & 3 & 0 & 2 & -1 \\ & & 12 & 48 & \\ \hline & 3 & 12 & & \end{array}$$

6. Repeat Step 4 for all columns. In this example, you get

$$\begin{array}{r|rrrr} 4 & 3 & 0 & 2 & -1 \\ & & 12 & 48 & 200 \\ \hline & 3 & 12 & 50 & 199 \end{array}$$

7. The bottom row of numbers reveals the answer along with the remainder. In this case, the numbers **3 12 50 199** indicate an answer of
   $3x^2 + 12x + 50$, **with remainder 199 or** $3x^2 + 12x + 50 + 199/(x - 4)$

Note: The answer will always have degree one less than the dividend.

Example 4: Using synthetic division, find the quotient and remainder

a) $f(x) = \dfrac{x^4 + 2x^3 - 6x^2 - 2}{x - 1}$

b) $x^5 + 32 \div x + 2$

Example 3.2.1 Page 261: Reading

Homework Practice Problems from the Text
Exercise 3.2.1 Page 265 #7 – 20

The Factor Theorem:

For a polynomial $f(x)$, if $f(c) = 0$, then $x - c$ is a factor of $f(x)$.

Note: $x - c$ is a factor of $f(x)$ means, the remainder when $f(x)$ divided by $x - c$ is **0**

Example 5: Let $f(x) = x^3 - 2x^2$.

$f(2) = 0$, so by the **Factor Theorem,** $x - 2$ is a factor of $f(x)$

Example 6: Let $f(x) = x^3 + 2x^2 - 5x - 6$

a) Use **long division** to determine whether $x + 3$ and $x - 3$ are factors of $f(x)$.

b) Use **The Factor Theorem** to determine whether $x + 3$ and $x - 3$ are factors of $f(x)$.

c) Use **synthetic division** to determine whether $x + 3$ and $x - 3$ are factors of $f(x)$.

The Remainder Theorem:

If $f(x) = (x - c)Q(x) + R$, then $f(c) = R$. That is, the **remainder** when dividing the polynomial $f(x)$ by $x - c$ is the same as the **value** of the **function** evaluated at $x = c$.

Example 7: Using the **Remainder Theorem**, find the remainder when $f(x) = x^3 + 2x^2 - 5x - 6$ is divided by**:**

a) $x + 2$

b) $x - 1$

Example 3.2.2 Page 362: Reading

Example 8: Decide whether the numbers $\mathbf{-3,\ 2,}$ are zeros of the polynomial

$f(x) = 3x^3 + 5x^2 - 6x + 18$; use both **Synthetic Division** and the **Remainder Theorem**

Example 9: Factor the polynomial $\boldsymbol{f(x) = x^3 + 5x^2 - 2x - 24}$ and solve the equation $\boldsymbol{f(x) = 0.}$

Solution:

1) First, list all integral factors of **– 24**: which are $\mathbf{\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 8,\ \pm 12,\ \pm 24}$
2) Next, check if any of these factor is a zero of $f(x)$

   Check for: $\boldsymbol{f(\pm 1) =?,\ f(\pm 2) =?,\ etc.}$
3) Finally using the result of 2) and division of polynomials factor $f(x)$

Example 10: Factor the polynomial $\boldsymbol{f(x) = x^4 + 2x^3 - 25x^2 - 50x}$ completely

Example 11: Factor the polynomial $\boldsymbol{f(x) = x^4 - 8x^4 - 33}$ completely

Example 12: Factor the polynomial $\boldsymbol{f(x) = x^4 - 5x^3 + 20x - 16}$ completely

Example 12: Solve $\boldsymbol{x^3 + 4x^2 + 25x - 100 = 0}$

OER: West Texas A&M University

Tutorial 37: **Synthetic Division and the Remainder and Factor Theorems**

OER: West Texas A&M University on zeros of polynomial functions

Tutorial 38: **Zeros of Polynomial Functions, Part I**

Tutorial 39: **Zeros of Polynomial Functions, Part II**

Homework Practice Problems from the Text

Exercise 3.2.1 Page 265 #21 – 46 (odd numbers)

Examples YouTube videos

- Polynomial Long Division 1: https://www.youtube.com/watch?v=4u8_AMacu-Y
- Polynomial long Division 2: https://www.youtube.com/watch?v=FXgV9ySNusc
- Synthetic Division 1: https://www.youtube.com/watch?v=1byR9UEQJN0